

Continuous Probability Distributions

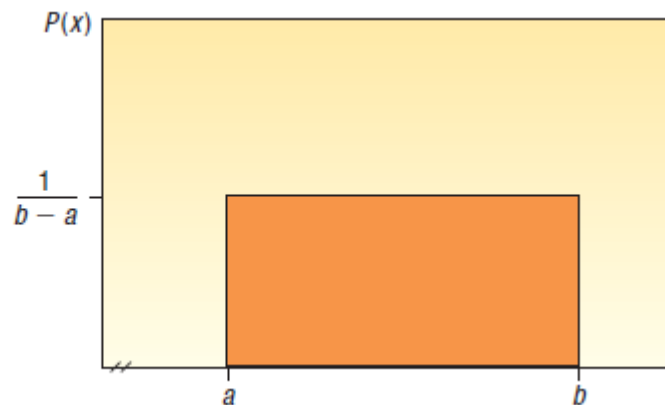
Chapter 7

Learning Objectives

- LO7-1 Describe the uniform probability distribution and use it to calculate probabilities
- LO7-2 Describe the characteristics of a normal probability distribution
- LO7-3 Describe the standard normal probability distribution and use it to calculate probabilities
- LO7-4 Approximate the binomial probability distribution using the standard normal probability distribution to calculate probabilities
- LO7-5 Describe the exponential probability distribution and use it to calculate probabilities

Uniform Distribution

- ▶ The uniform distribution characteristics
 - ▶ It is rectangular in shape
 - ▶ The mean and the median are equal
 - ▶ It is completely described by its minimum value a and its maximum value b



Uniform Distribution Formulas

- ▶ The mean and standard deviation of a uniform distribution are computed as follows

MEAN OF THE UNIFORM DISTRIBUTION

$$\mu = \frac{a + b}{2}$$

[7-1]

**STANDARD DEVIATION
OF THE UNIFORM DISTRIBUTION**

$$\sigma = \sqrt{\frac{(b - a)^2}{12}}$$

[7-2]

- ▶ The following equation describes the region from a to b

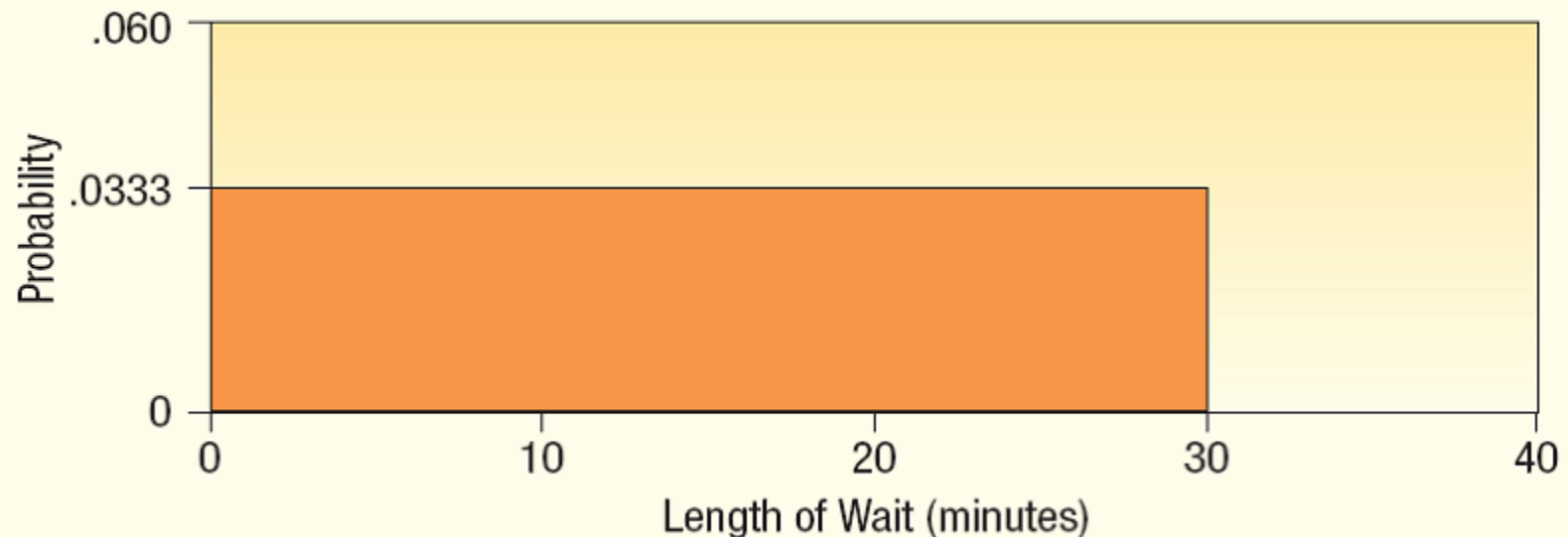
UNIFORM DISTRIBUTION

$$P(x) = \frac{1}{b - a} \text{ if } a \leq x \leq b \text{ and } 0 \text{ elsewhere}$$

[7-3]

Uniform Distribution Example

Southwest Arizona State University provides bus service to students while they are on campus. A bus arrives at the North Main Street and College Drive stop every 30 minutes between 6 a.m. and 11 p.m. during weekdays. Students arrive at the bus stop at random times. The time that a student waits is uniformly distributed from 0 to 30 minutes.



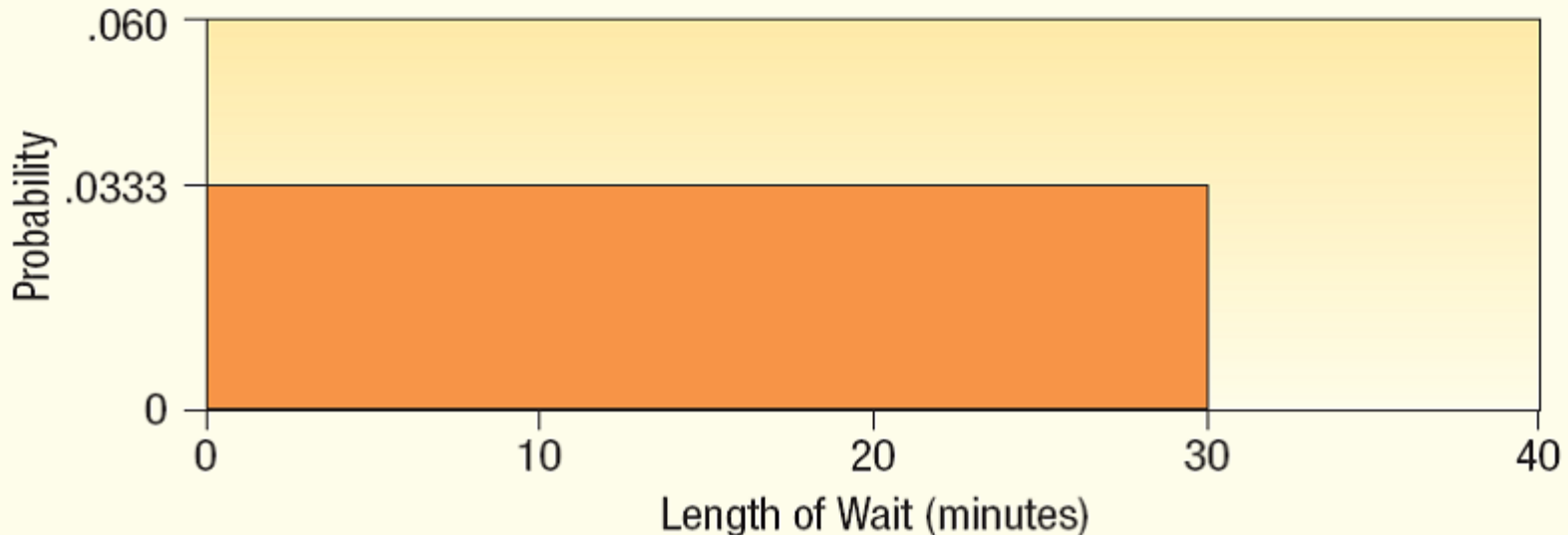
Uniform Distribution Example

The area of the uniform distribution is found by multiplying height *base

$$\text{Area} = \frac{1}{30-0} (30-0) = 1.00$$

$$\text{The mean is } \mu = \frac{a+b}{2} = \frac{0+30}{2} = 15$$

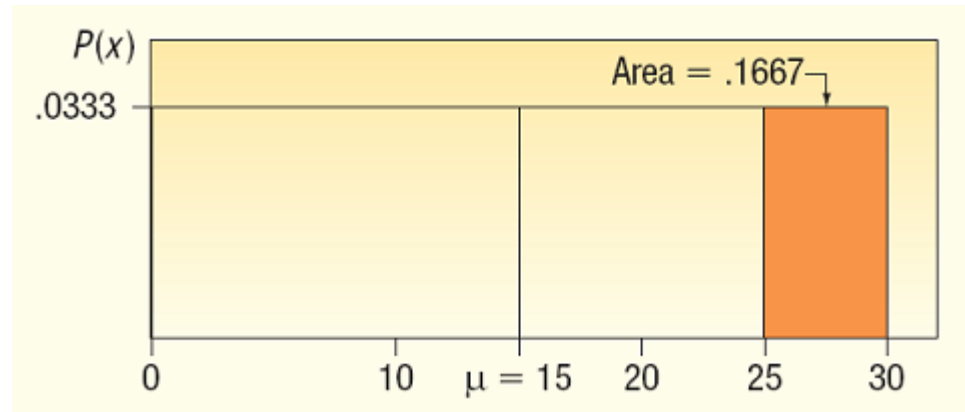
$$\text{The standard deviation is } \sigma = \sqrt{\frac{(b-a)^2}{12}} = \sqrt{\frac{(30-0)^2}{12}} = 8.66$$



Uniform Distribution Example

To find the probability that a student will wait more than 25 minutes, find the area between 25 and 30 minutes.

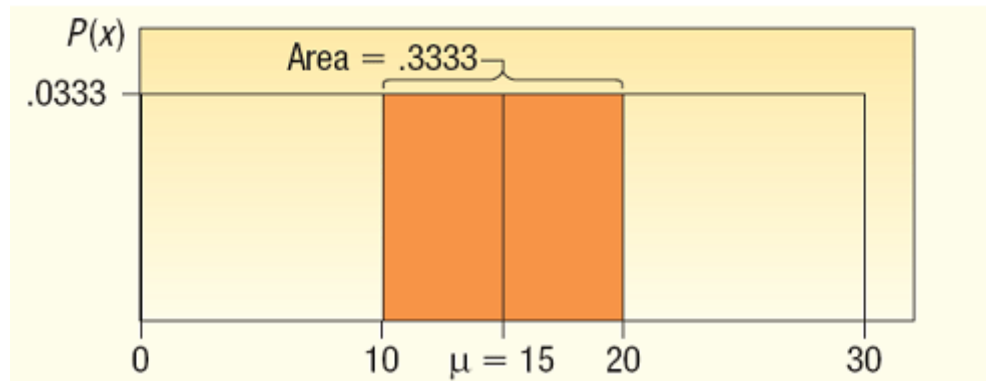
$$P(25 < \text{wait time} < 30) = (\text{height})(\text{base}) = \frac{1}{(30-0)} (5) = .1667$$



Uniform Distribution Example

To find the probability that a student will wait between 10 and 20 minutes, find the area between 10 and 20 minutes.

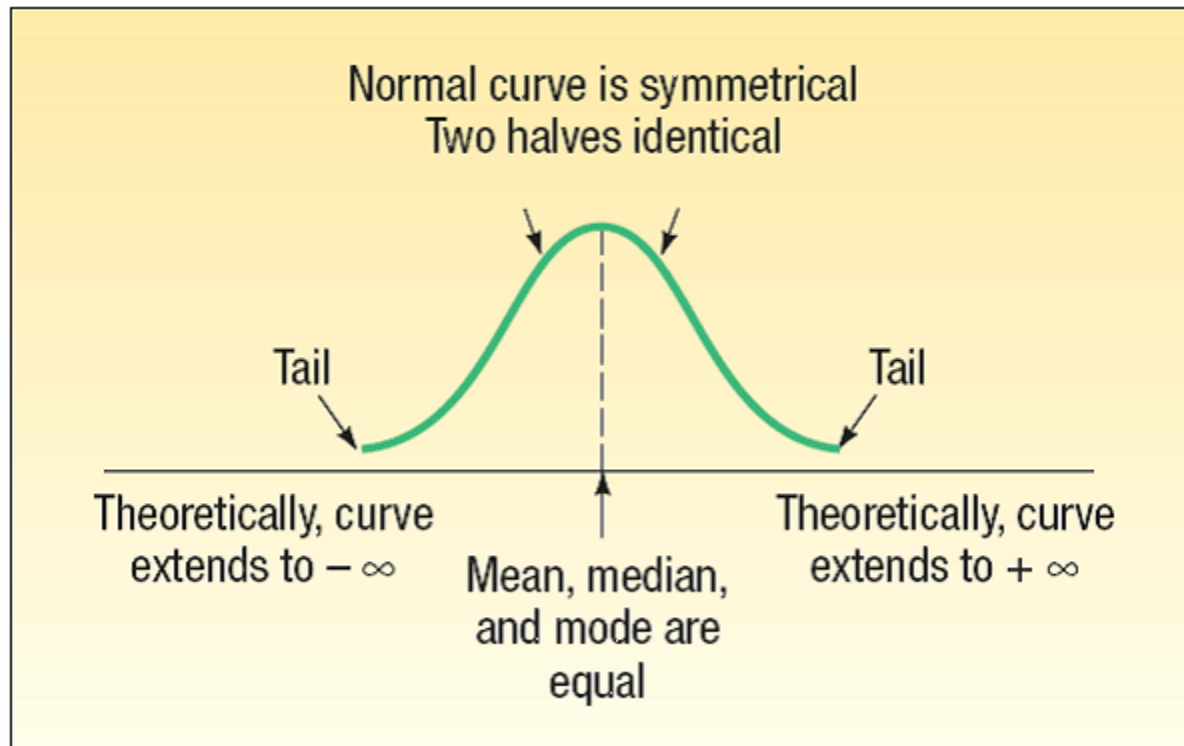
$$P(10 < \text{wait time} < 20) = (\text{height})(\text{base}) = \frac{1}{(30-0)} (10) = .3333$$



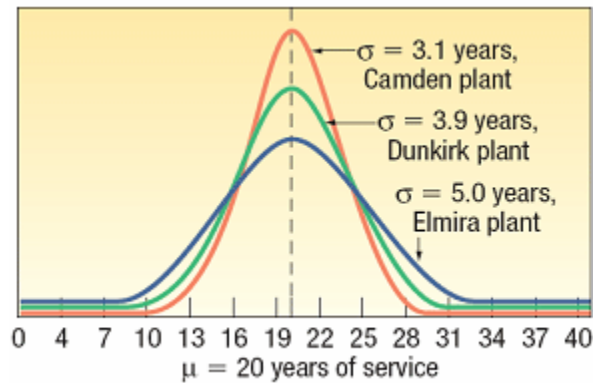
Normal Probability Distribution

- ▶ The normal probability distribution is a continuous distribution with the following characteristics
 - ▶ It is bell-shaped and has a single peak at the center of the distribution
 - ▶ The distribution is symmetric
 - ▶ It is asymptotic, meaning the curve approaches but never touches the X-axis
 - ▶ It is completely described by its mean and standard deviation
- ▶ There is a family of normal probability distributions
 - ▶ Another normal probability distribution is created when either the mean or the standard deviation changes

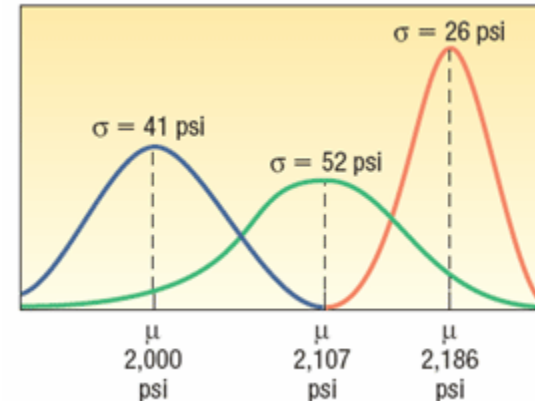
The Normal Curve



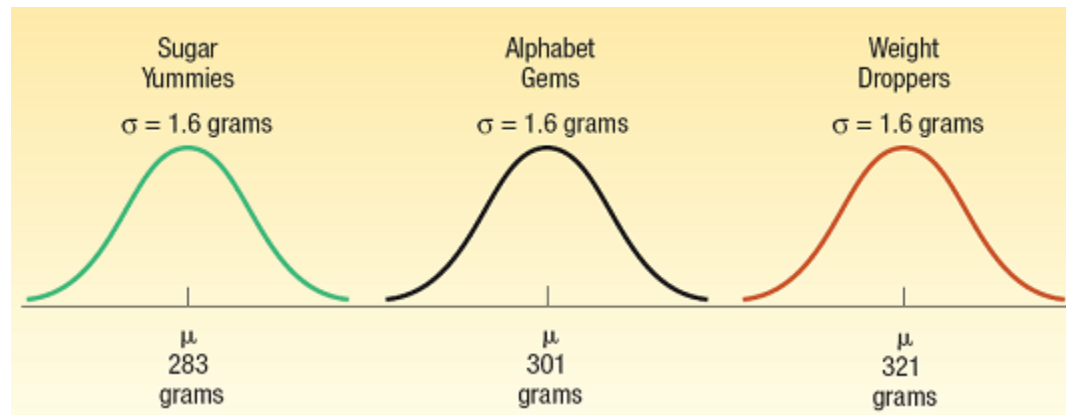
Family of Normal Probability Distributions



Equal Means and Different Standard Deviations



Different Means and Standard Deviations



Different Means and Equal Standard Deviations

Standard Normal Probability Distribution

- ▶ The standard normal probability distribution is a particular normal distribution
- ▶ It has a mean of 0 and a standard deviation of 1

zVALUE The signed distance between a selected value, designated x , and the mean, μ , divided by the standard deviation, σ .

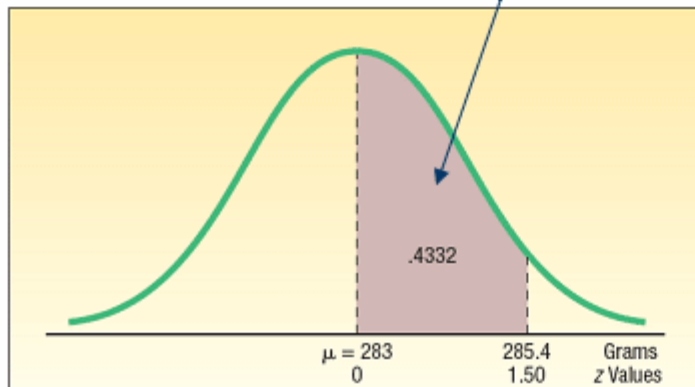
- ▶ Any normal probability distribution can be converted to the standard normal probability distribution with the following formula

$$Z = \frac{X - \mu}{\sigma}$$

Areas Under the Normal Curve

- Here is a portion of the “z” Table

z	0.00	0.01	0.02	0.03	0.04	0.05	...
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	
.							
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Standard Normal Probability Example

In recent years a new type of taxi service has evolved in more than 300 cities world-wide, where the customer is connected directly with a driver via a smartphone. It uses the Uber mobile app, which allows customers with a smartphone to submit a trip request which is then routed to a Uber driver who picks up the customer and takes the customer to the desired location. No cash is involved, the payment for the transaction is handled via a digital payment. Suppose the weekly income of Uber drivers follows the normal probability distribution with a mean of \$1,000 and a standard deviation of \$100.

What is the z-value of income for a driver who earns \$1,100?

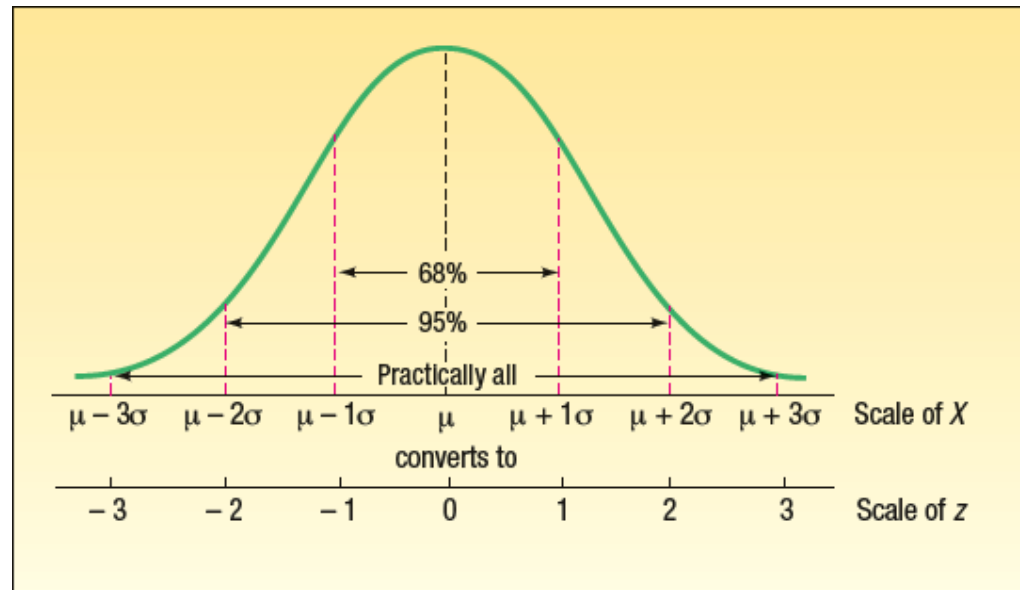
$$Z = \frac{x - \mu}{\sigma} = \frac{\$1,100 - \$1,000}{\$100} = 1.00$$

What is the z-value of income for a driver who earns \$900?

$$Z = \frac{x - \mu}{\sigma} = \frac{\$900 - \$1,000}{\$100} = -1.00$$

Regardless of whether z is +1 or -1, the area under the curve is .3413

The Empirical Rule



To verify the Empirical Rule:

z of 1.00 = .3413 so $.3413 * 2 = .6826$ or about 68%

z of 2.00 = .4772 so $.4772 * 2 = .9544$ or about 95%

z of 3.00 = .4987 so $.4987 * 2 = .9974$ or about 99.7%

The Empirical Rule Example

As part of its quality assurance program, the Autolite Battery Company conducts tests on battery life. For a particular D-cell alkaline battery, the mean life is 19 hours. The useful life of the battery follows a normal distribution with a standard deviation of 1.2 hours.

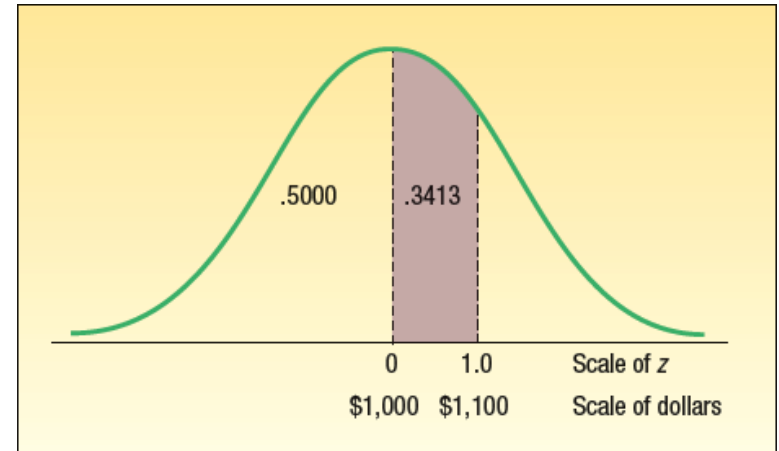
1. About 68% of the batteries failed between what two values?
 $\mu \pm 1$ standard deviation; $19 \pm 1(1.2)$ hours;
About 68% of batteries will fail between 17.8 and 20.2 hours.
2. About 95% of the batteries failed between what two values?
 $\mu \pm 2$ standard deviations; $19 \pm 2(1.2)$ hours;
About 95% of batteries will fail between 16.6 and 21.4 hours.
3. Virtually all of the batteries failed between what two values?
 $\mu \pm 3$ standard deviations; $19 \pm 3(1.2)$ hours;
Practically all of the batteries will fail between 15.4 and 22.6 hours.

Finding Areas under the Normal Curve

What is the z-value of income for a driver who earns \$1,100?

$$Z = \frac{x - \mu}{\sigma} = \frac{\$1,100 - \$1,000}{\$100} = 1.00$$

z	0.00	0.01	0.02
⋮	⋮	⋮	⋮
0.7	.2580	.2611	.2642
0.8	.2881	.2910	.2939
0.9	.3159	.3186	.3212
1.0	.3413	.3438	.3461
1.1	.3643	.3665	.3686
⋮	⋮	⋮	⋮



Using the weekly incomes of Uber drivers:

$$P(\$1,000 < \text{weekly income} < \$1,100) = .3413$$

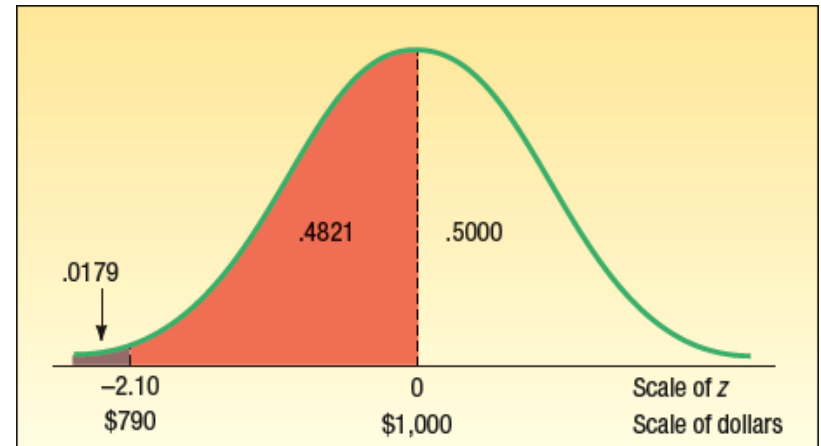
$$P(\text{weekly income} < \$1,100) = .3413 + .5000 = .8413$$

Finding Areas under the Normal Curve

What is the z-value of income for a driver who earns \$790?

$$Z = \frac{x - \mu}{\sigma} = \frac{\$790 - \$1,000}{\$100} = -2.10$$

z	0.00	0.01	0.02
⋮	⋮	⋮	⋮
2.0	.4772	.4778	.4783
2.1	.4821	.4826	.4830
2.2	.4861	.4864	.4868
2.3	.4893	.4896	.4898
⋮	⋮	⋮	⋮



Using the weekly incomes of Uber drivers:

$$P(\$790 < \text{weekly income} < \$1,000) = .4821$$

$$P(\text{weekly income} < \$790) = .5000 - .4821 = .0179$$

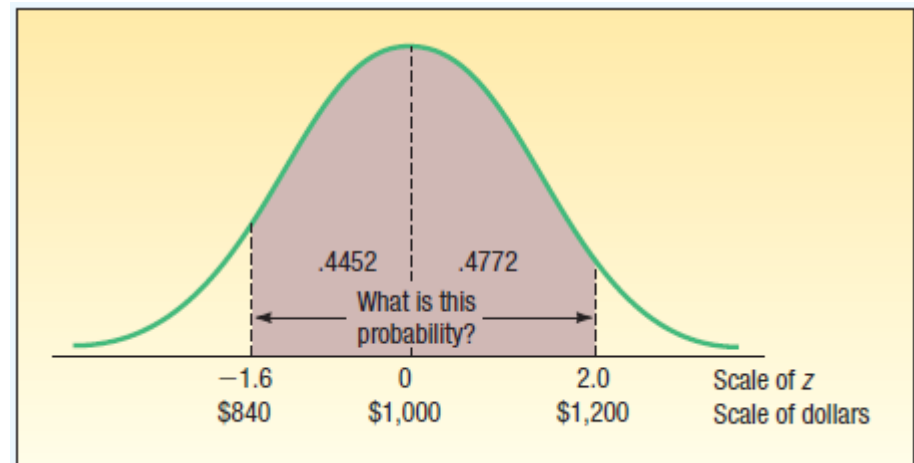
Finding Areas Under the Normal Curve

What is the z-value of income for a driver who earns \$840?

$$Z = \frac{x - \mu}{\sigma} = \frac{\$840 - \$1,000}{\$100} = -1.60$$

What is the z-value of income for a driver who earns \$1,200?

$$Z = \frac{x - \mu}{\sigma} = \frac{\$1,200 - \$1,000}{\$100} = 2.00$$



Using the weekly incomes of Uber drivers:

$$P(\$840 < \text{weekly income} < \$1,000) = .4452$$

$$P(\$1,000 < \text{weekly income} < \$1,200) = .4772$$

$$P(\$840 < \text{weekly income} < \$1,200) = .4452 + .4772 = .9224$$

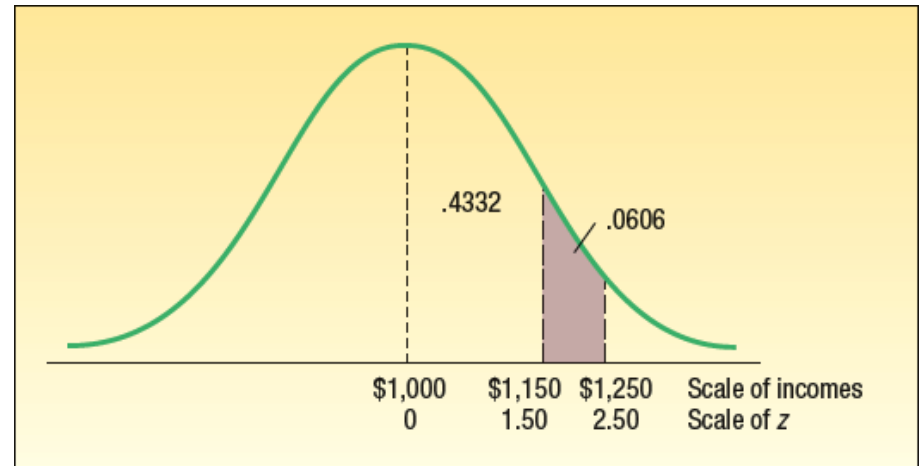
Finding Areas Under the Normal Curve

What is the z-value of income for a driver who earns \$1,250?

$$Z = \frac{x - \mu}{\sigma} = \frac{\$1,250 - \$1,000}{\$100} = 2.50$$

What is the z-value of income for a driver who earns \$1,150?

$$Z = \frac{x - \mu}{\sigma} = \frac{\$1,150 - \$1,000}{\$100} = 1.50$$



Using the weekly incomes of Uber drivers:

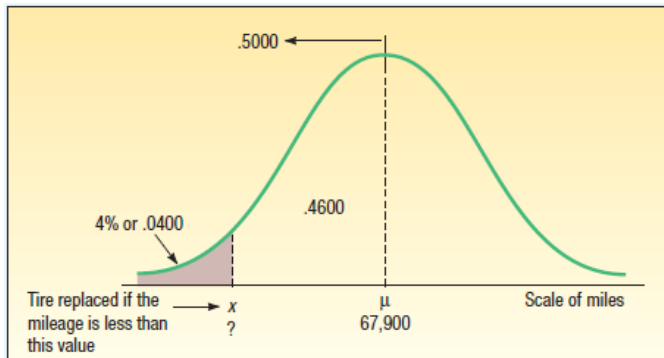
$$P(\$1,000 < \text{weekly income} < \$1,250) = .4938$$

$$P(\$1,000 < \text{weekly income} < \$1,150) = .4332$$

$$P(\$1,150 < \text{weekly income} < \$1,250) = .4938 - .4332 = .0606$$

Finding a Value for x Using z

Layton Tire and Rubber Company wishes to set a minimum mileage guarantee on its new MX100 tire. Tests reveal the mean mileage is 67,900 with a standard deviation of 2,050 miles and that the distribution follows the normal distribution. Let x represent the minimum guaranteed mileage and use the formula for z to solve so that no more than 4% of tires need to be replaced.



z ...	.03	.04	.05	.06
⋮	⋮	⋮	⋮	⋮
1.5	.4370	.4382	.4394	.4406
1.6	.4484	.4495	.4505	.4515
1.7	.4582	.4591	.4599	.4608
1.8	.4664	.4671	.4678	.4686

$$z = \frac{x - \mu}{\sigma} = \frac{x - 67,900}{2,050} \text{ and from the table we find } z = -1.75$$

$$\text{so } -1.75 = \frac{x - 67,900}{2,050} \text{ therefore, } x = 64,312 \text{ miles}$$

Exercise (1 / 4)

Exercise 4: According to the Insurance Institute of America, a family of four spends between \$400 and \$3,800 per year on all types of insurance. Suppose the money spent is uniformly distributed between these amounts.

- a. What is the mean amount spent on insurance?
- b. What is the standard deviation of the amount spent?
- c. If we select a family at random, what is the probability they spend less than \$2,000 per year on insurance per year?
- d. What is the probability a family spends more than \$3,000 per year?

Exercise (2/4)

Exercise 9: The mean of a normal probability distribution is 500; the standard deviation is 10.

- About 68 percent of the observations lie between what two values?
- About 95 percent of the observations lie between what two values?
- Practically all of the observations lie between what two values?

Exercise 13: A normal population has a mean of 20.0 and a standard deviation of 4.0.

- Compute the *z value* associated with 25.0.
- What proportion of the population is between 20.0 and 25.0?
- What proportion of the population is less than 18.0?

<i>z</i>	0.00	0.01	0.02	0.03	0.04	0.05
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265

Exercise (3/4)

Exercise 17: A normal distribution has a mean of 50 and a standard deviation of 4.

- Compute the probability of a value between 44.0 and 55.0.
- Compute the probability of a value greater than 55.0.
- Compute the probability of a value between 52.0 and 55.0.

<i>z</i>	0.00	0.01	0.02	0.03	0.04	0.05
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394

Exercise (4/4)

Exercise 26: The SAT Reasoning Test (formerly called the Scholastic Aptitude Test) is perhaps the most widely used standardized test for college admissions in the United States. Scores are based on a normal distribution with a mean of 1500 and a standard deviation of 300. Clinton College would like to offer an honors scholarship to students who score in the top 10 percent of this test.

- What is the minimum score that qualifies for the scholarship?

<i>z</i>	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306

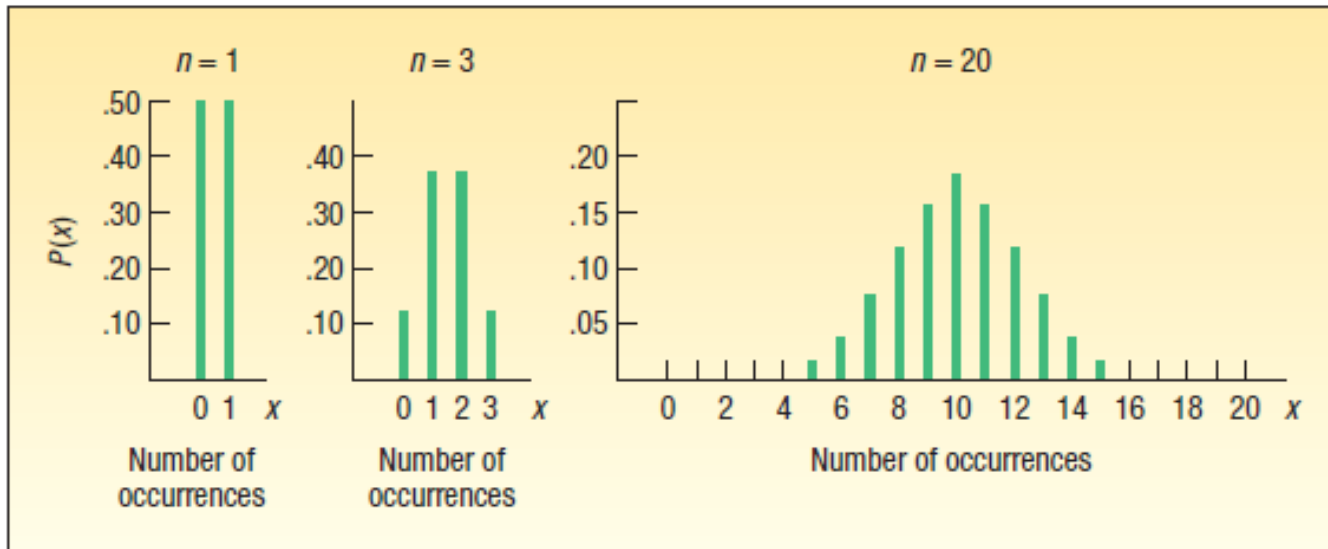
Approximate a Binomial Distribution

- ▶ The normal probability distribution can approximate a binomial distribution under certain conditions
 - ▶ $n\pi$ and $n(1-\pi)$ must both be at least 5
 - ▶ n is the number of observations
 - ▶ π is the probability of a success
- ▶ The four conditions for a binomial probability distribution are
 - ▶ There are only two possible outcomes
 - ▶ π (p_i) remains the same from trial to trial
 - ▶ The trials are independent
 - ▶ The distribution results from a count of the number of successes in a fixed number of trials

Used for large values of n , as n increases, a binomial gets closer to a normal distribution

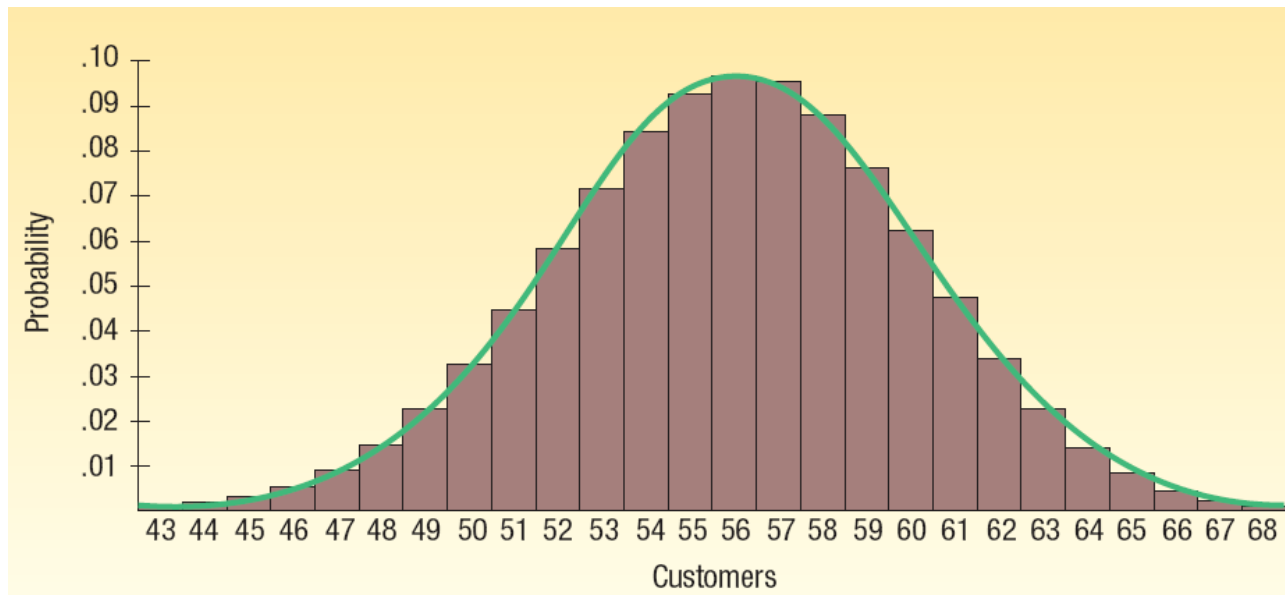
Approximate a Binomial Distribution

- Notice how the distribution approximates the shape of a normal distribution as n becomes larger



Approximate a Binomial Distribution

CONTINUITY CORRECTION FACTOR The value .5 subtracted or added, depending on the question, to a selected value when a discrete probability distribution is approximated by a continuous probability distribution.



How to Apply the Correction Factor

- ▶ Only four cases may arise
 - ▶ For the probability *at least* x occur,
use the area above $(x - .5)$
 - ▶ For the probability that *more than* x occur,
use the area above $(x + .5)$
 - ▶ For the probability that x or *fewer* occur,
use the area below $(x + .5)$
 - ▶ For the probability that *fewer than* x occur,
use the area below $(x - .5)$

Correction Factor Example

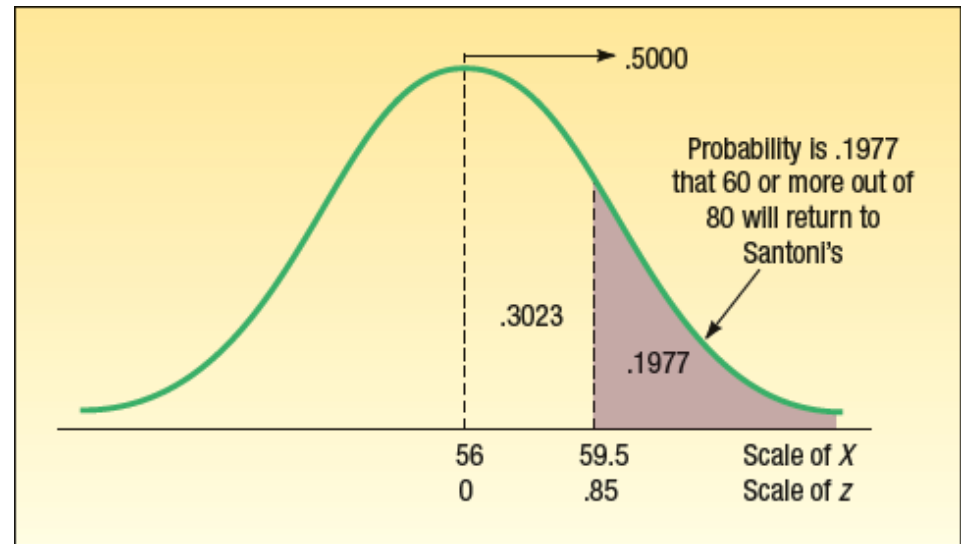
Suppose the management of the Santoni Pizza Restaurant found that 70% of its new customers return for another meal. For a week in which 80 new customers dined at Santoni's, what is the probability that 60 or more will return for another meal?

$$\mu = n\pi = (80)(.70) = 56$$

$$\sigma^2 = n\pi(1-\pi) = (80)(.70)(1-.70) = 16.8$$

$$\sigma = \sqrt{16.8} = 4.10$$

$$z = \frac{x - \mu}{\sigma} = \frac{59.5 - 56}{4.10} = 0.85$$



$$P(\text{customers} > 59.5) = .5000 - .3023 = .1977$$

$$\sigma = \sqrt{9.375} = 3.0619$$

Exercise

Exercise 31: Assume a binomial probability distribution with $n = 50$ and $\Phi = 0.25$

Compute the following:

- The mean and standard deviation of the random variable.
- The probability that X is 15 or more.
- The probability that X is 10 or less.

$$\mu = n\pi = 50(0.25) = 12.5 \quad \sigma^2 = n\pi(1 - \pi) = 12.5(1 - 0.25) = 9.375$$

$$\sigma = \sqrt{9.375} = 3.0619$$

b. $(14.5 - 12.5)/3.0619 = 0.65$, the area is 0.2422, then $0.5000 - 0.2422 = 0.2578$

For the probability at least x occur, use the area above $(x - .5)$

c. $(10.5 - 12.5)/3.0619 = -0.65$. The area is 0.2422. Then $0.5000 - 0.2422 = 0.2578$

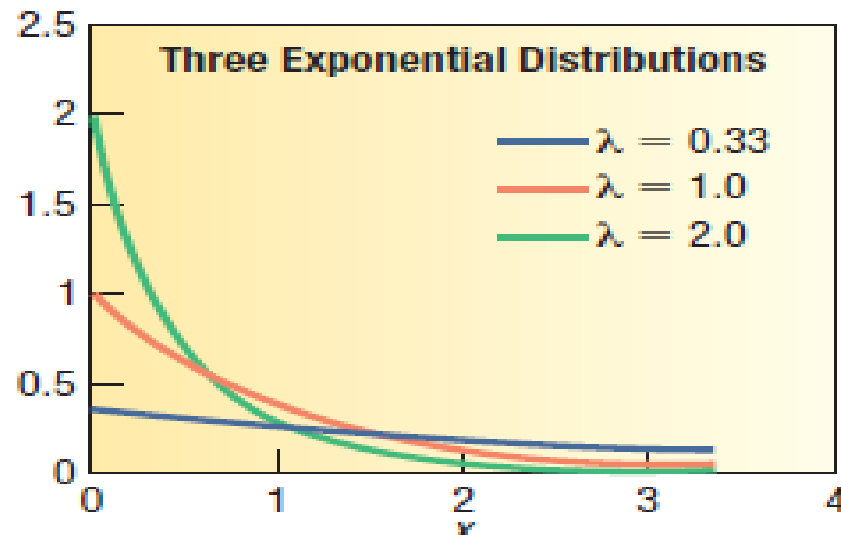
For the probability that x or fewer occur, use the area below $(x + .5)$

Exponential Probability Distribution

- ▶ The exponential probability distribution describes times between events in a sequence
- ▶ The actions occur independently at a constant rate per unit of time or length
- ▶ It is non-negative, is positively skewed, declines steadily to the right, and is asymptotic
- ▶ Examples of situations using the exponential distributions
 - ▶ The service time for customers at the information desk at Dallas Public Library
 - ▶ The time until the next phone call arrives in a customer service center

The Family of Exponential Distributions

- ▶ There is not just one, exponential distribution, but a family of them
- ▶ λ , lambda, is the rate parameter
- ▶ The lower the rate parameter, the “less skewed” the shape of the distribution



Exponential Distribution Formulas

- ▶ This formula describes the exponential distribution

EXPONENTIAL DISTRIBUTION

$$P(x) = \lambda e^{-\lambda x}$$

(7-6)

- ▶ The area under the curve is given by this formula

FINDING A PROBABILITY

USING THE

EXPONENTIAL DISTRIBUTION

$$P(\text{Arrival time} < x) = 1 - e^{-\lambda x}$$

(7-7)

- ▶ Both the mean and the standard deviation are

$$\mu = \frac{1}{\lambda} \quad \text{and} \quad \sigma = \frac{1}{\lambda}$$

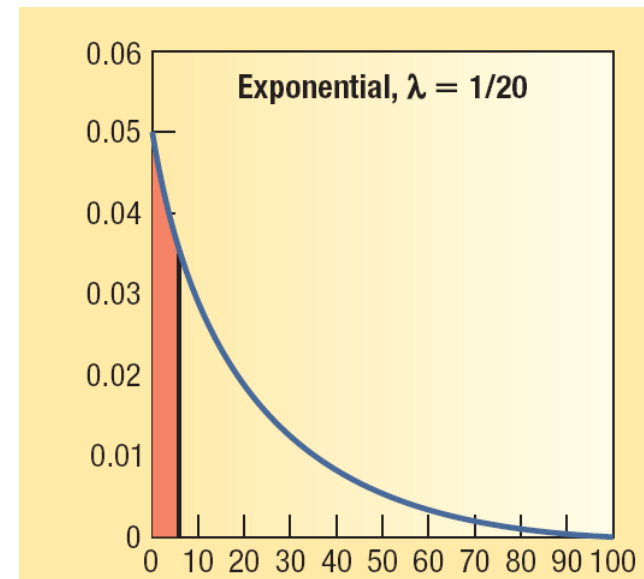
Exponential Distribution Example

Orders for prescriptions arrive at a pharmacy website according to an exponential probability distribution at a mean of one every 20 seconds.

Find the probability the next order arrives in less than 5 seconds.

$$P(\text{Arrival time} < x) = 1 - e^{-\lambda x}$$

$$\begin{aligned} P(\text{Arrival time} < 5) &= 1 - e^{-\frac{1}{20}(5)} \\ &= 1 - e^{-0.25} \\ &= 1 - .7788 = .2212 \end{aligned}$$



Exponential Distribution Example

Orders for prescriptions arrive at a pharmacy website according to an exponential probability distribution at a mean of one every 20 seconds.

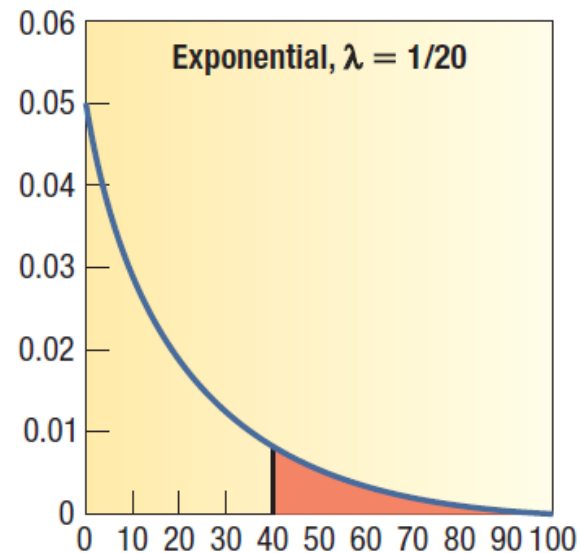
Find the probability the next order arrives in more than 40 seconds.

$$P(\text{Arrival time} < x) = 1 - e^{-\lambda x}$$

$$P(\text{Arrival time} > x) = e^{-\lambda x}$$

$$P(\text{Arrival time} > 40) = e^{-\frac{1}{20}(40)} = .1353$$

Complement rule



Exercise (1 / 1):

Exercise 38: The lifetime of plasma and LCD TV sets follows an exponential distribution with a mean of 100,000 hours. Compute the probability a television set:

a. Fails in less than 10,000 hours.

$$P(\text{Arrival time} < x) = 1 - e^{-\lambda x}$$

b. Lasts more than 120,000 hours.

$$P(\text{Arrival time} > x) = e^{-\lambda x}$$

c. Fails between 60,000 and 100,000 hours of use.

d. Find the 90th percentile. So 10 percent of the TV sets last more than what length of time?

$$P(\text{Arrival time} > x) = e^{-\lambda x}$$

$$\lambda = 1 / 100,000$$

$$P(\text{Arrival time} > x) = e^{-\lambda x}$$



$$\ln[p(x)] = -\lambda x$$



$$x = \ln[p(x)] / -\lambda$$

Excel: LN function